Geodätisches Institut – Universität Stuttgart

Bootstrap method and its application to the hypothesis testing in GPS mixed integer linear model

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Main Topics

1. Motivation
2. Brief review of statistical property of the GNSS carrier phase observables
3. Bootstrap methods for the confidence domains/ hypothesis tests
4. Conclusion and outlook

1. Motivation

- **The open problem to evaluate the statistical property of GPS carrier phase observables**

➡ **Ever since von Mises (1918) introduced the *von Mises normal distribution* on the circle, its importance has not been recognized by the data analysts;**

➡ **In practice, this fact is often ignored, for example, the statistical property of the *GPS carrier phase observations* are simply regarded as *Gauss-Laplace normal distribution*. And most of the existed validation and hypothesis tests (e.g. χ^2-test, *F*-test, *t*-test, and *ratio test* etc.) about the float and fixed solution of GPS mixed integer model are performed under this assumption;**

➡ **But according to our new research results (Cai, et al., 2007), the GPS carrier phase observables that are actually measured on the unit circle have been statistically validated to have a *von Mises* normal distribution;**

➡ **Therefore these validation and hypothesis testing procedures based on the Gauss normal distribution should be improved accordingly;**

➡ **Since the distributions of the statistics commonly used for inference on directional distributions are more complex than those arising in standard normal theory, *bootstrap methods* are particularly useful in the directional context.**

■ The observation equation of the GNSS carrier phase measurement

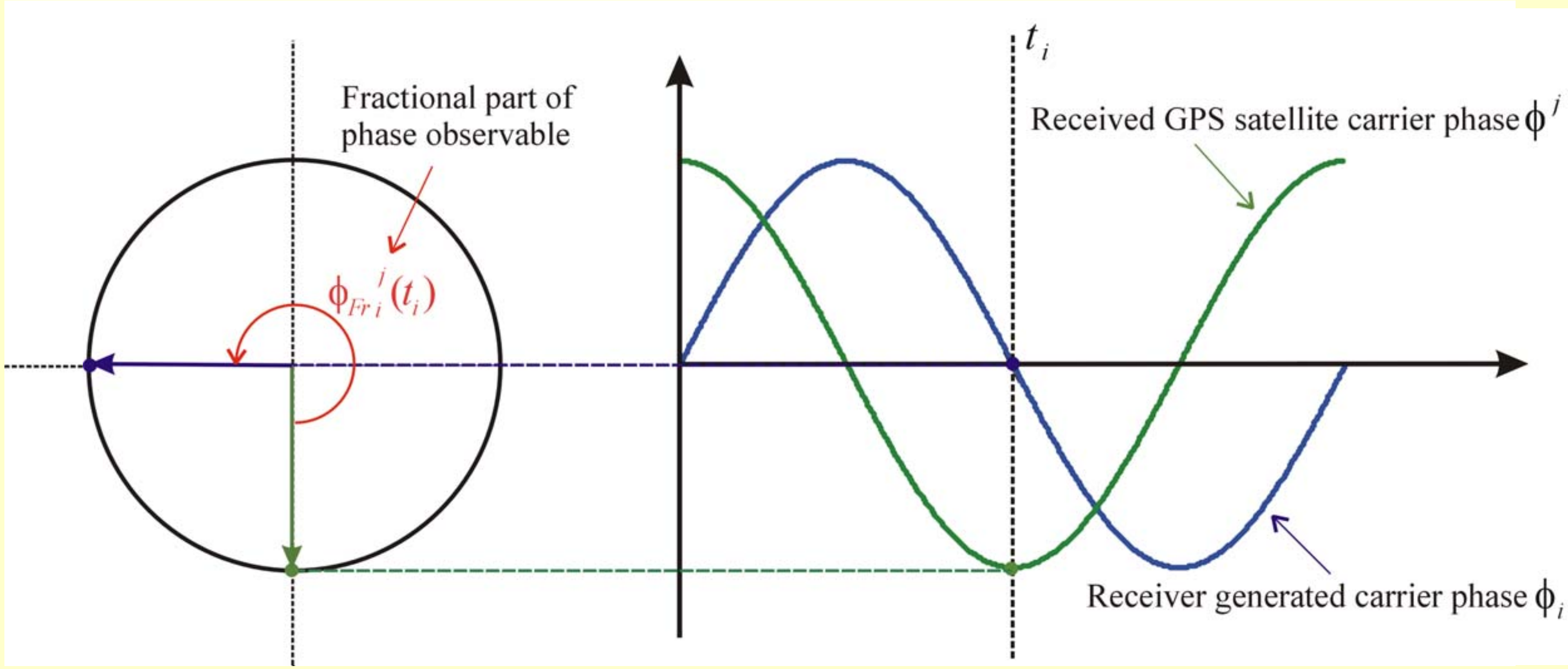
$$\varphi_k^p(t_k) = \varphi_{Fr\,k}^{\ \ p}(t_k) + N_k^p(t_k - t_0) =$$
$$= \frac{f}{c}\rho_k^p(t_k) + f[dT_k(t_k) - dt^p(t_k)] - \frac{f}{c}d_{I\,k}^{\ p}(t_k) +$$
$$+ \frac{f}{c}d_{T\,k}^{\ p}(t_k) + \frac{f}{c}d_{multi\,k}^{\ \ \ p}(t_k) - N_k^p(t_0) + e_{\varphi k}^{\ p}(t_k)$$

$\varphi_k^p(t_k)$ the carrier phase observation from satellite *p* and receiver *k* ;

$\varphi_{Fr\,k}^{\ \ p}(t_k)$ the fractional part of the phase difference (within the range: 0° to 360° as well as 0 to 1 circle);

$N_k^p(t_k - t_0)$ the sum of phase zero passes from start epoch t_0 to the time t_k (of the receiver observes)

Representation of the observations of GPS phase measurements



$$\phi_{Fr\,i}^{\;j}(t_i) = \phi_i\,(t_i) - \phi^j(t_i) = 0.5 - 0.75 = -0.25 \text{ (cycle)}$$

Since the fractional part is defined in [0, 1) or $[0, 2\pi)$

$$\phi_{Fr\,i}^{\;j}(t_i) = -0.25 + 1 = 0.75 \text{(cycle), or } = \frac{3}{2}\pi$$

2. Brief review of statistical property of the GNSS carrier phase observables

- **The von Mises distribution (1918) has the same important statistical role on the circle as the Gauss normal distribution on the line.**
- **The Fisher distribution (*Fisher* 1953) is of central important on the sphere for the three dimensional directional data.**
- **For the higher dimensional directional data the Langevin distribution is developed.**

The von Mises distribution:

- PDF of a circular random variable θ with von Mises distribution:

$$g(\theta;\mu_0,\kappa)=\frac{1}{2\pi I_0(\kappa)}e^{\kappa\cos(\theta-\mu_0)},\ -\pi\le\theta\le\pi,$$

$I_0(\kappa)$ - modified Bessel function.

the parameter μ_0 - mean direction

the parameter κ - concentration parameter

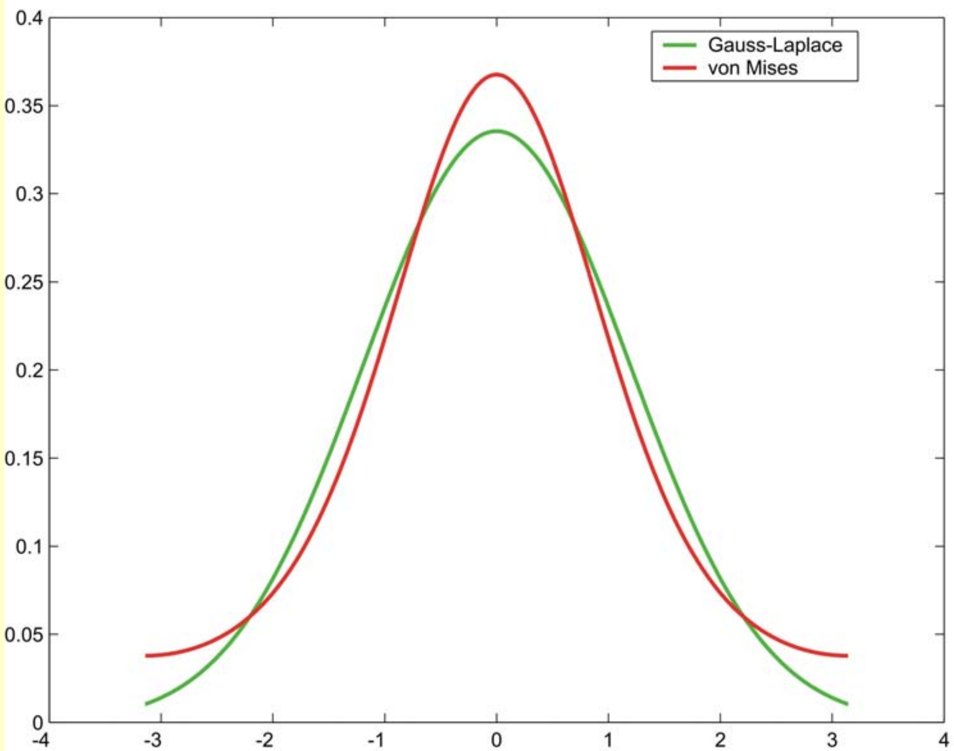
$$\hat{\kappa}=A^{-1}(\bar{R}),\ where\ A(\kappa)=I_1(\kappa)/I_0(\kappa).$$

- And the circualr variance V_0 is given by

$$V_0=1-\bar{R}.$$

- Note the PDF of the Gauss-Laplace normal distribution $\mathcal{N}(0,\sigma^2)$:

$$f(x;0,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}x^2}$$

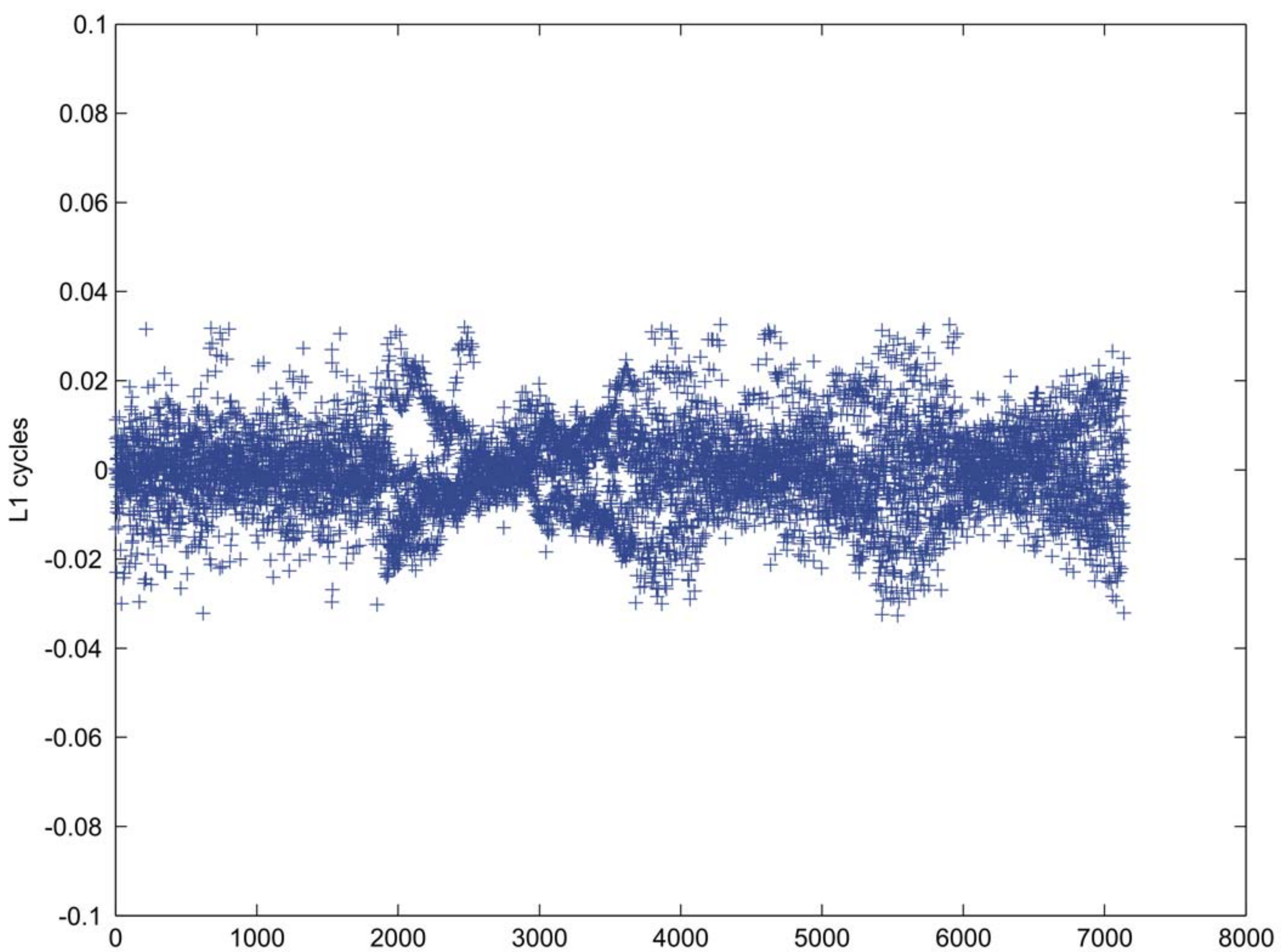


The density function of the von Mises (k=1.138) and Gauss-Laplace normal distribution (σ=1.189)

Test the statistical property of GPS carrier phase

GPS observation set:

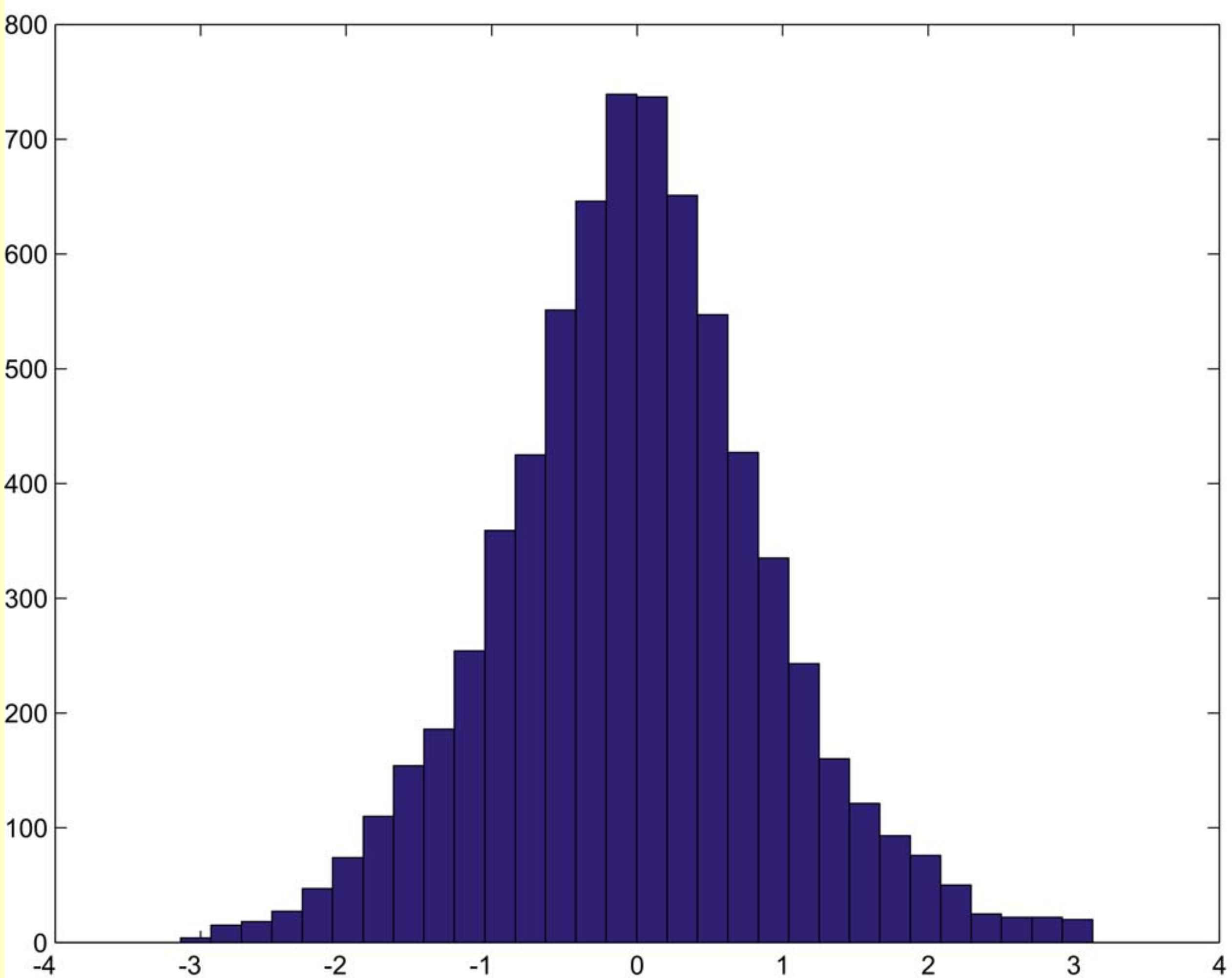
- **Short baselines test data: 2 hour observations with 20 second sampling rate at four baselines (2~3 km) in 2005.**
- **Phase baseline lengths were calculated using observations above 10º**
- **There are total 7198 L1 double difference phase observables, where these fractional phases are scaled to $[-\pi, \pi]$.**



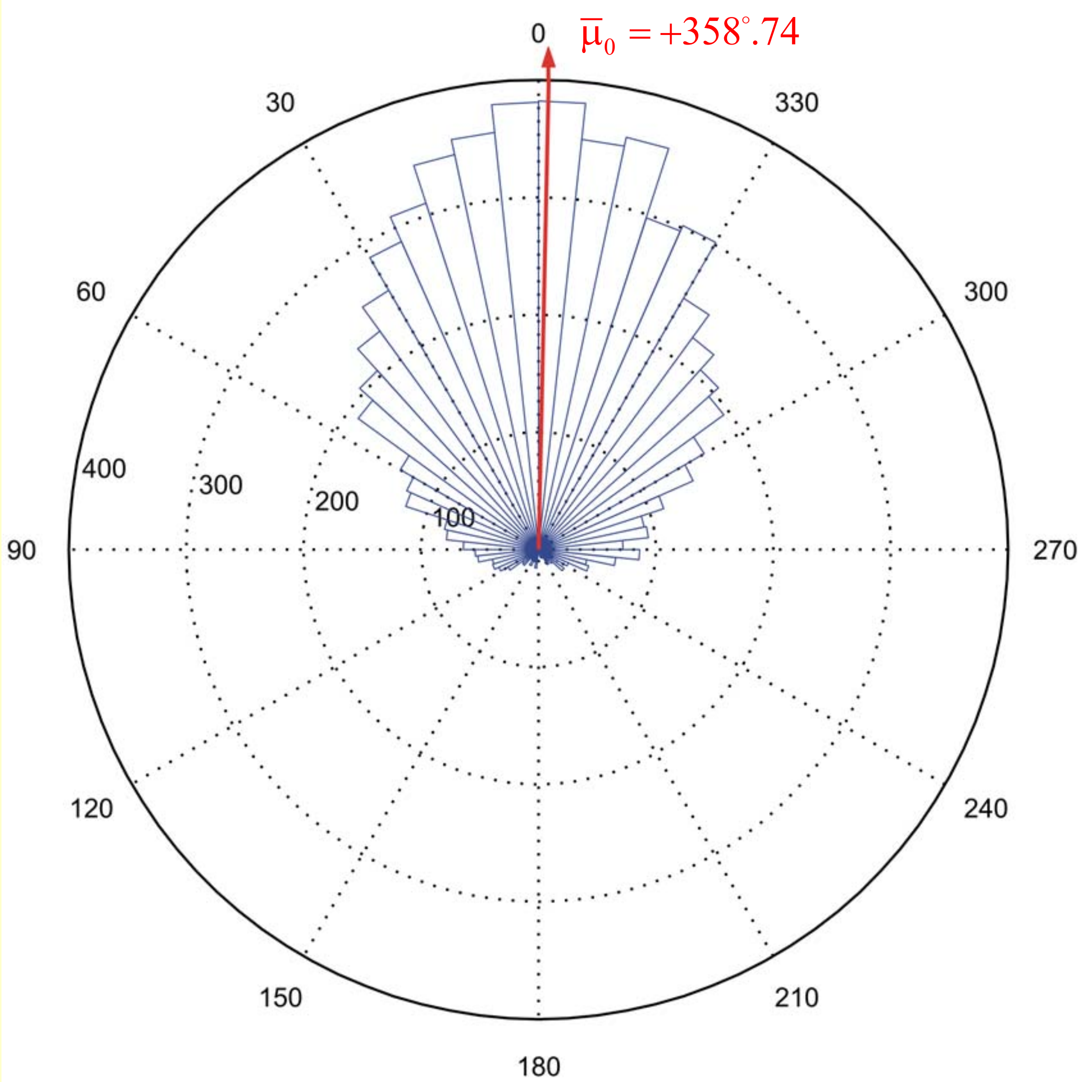
Example: L1 double difference phase observables with

σ=0.00973 (cycles) ~ 1.85 mm

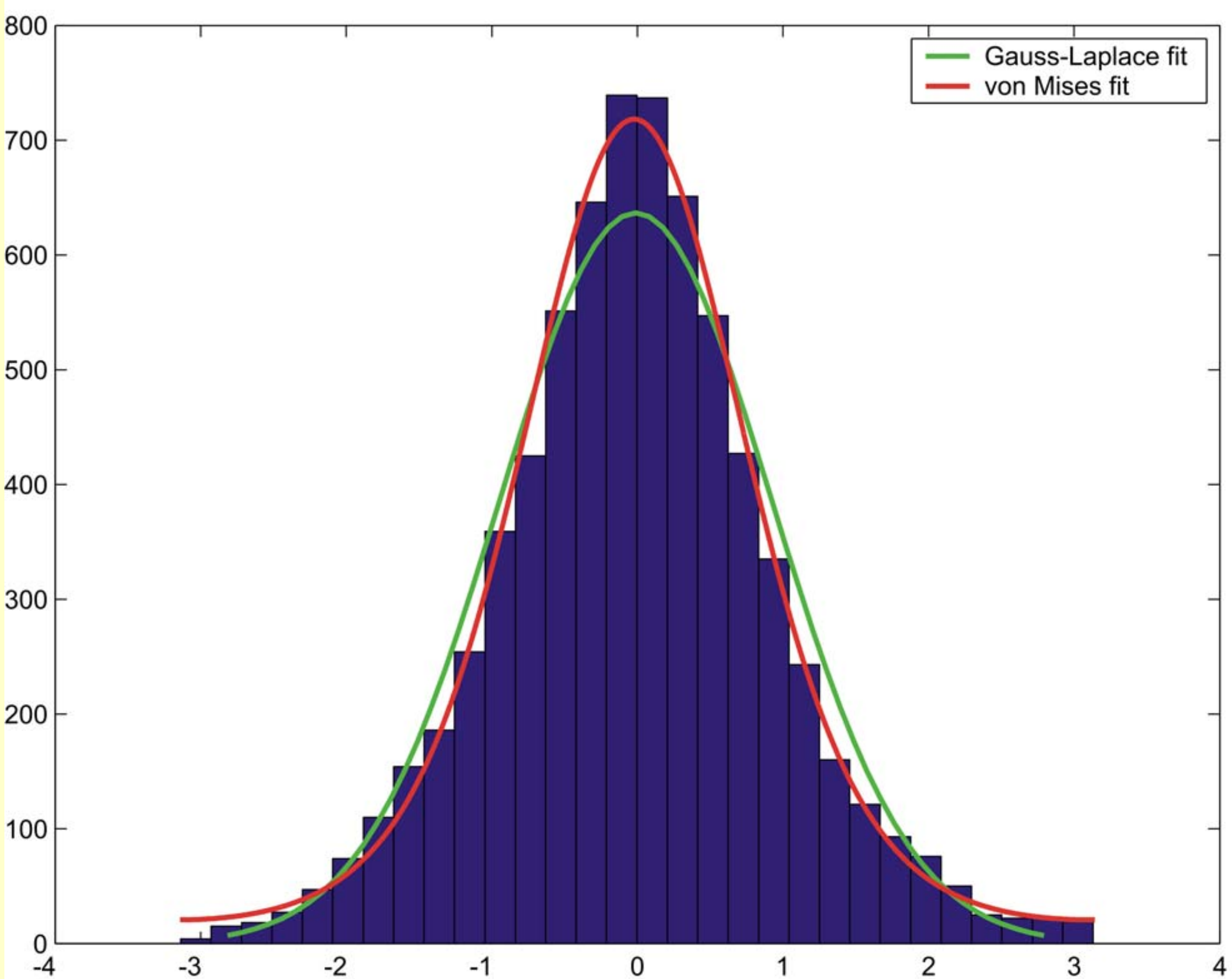
(7198 measurements *observed* on *four short baselines in 2005*)



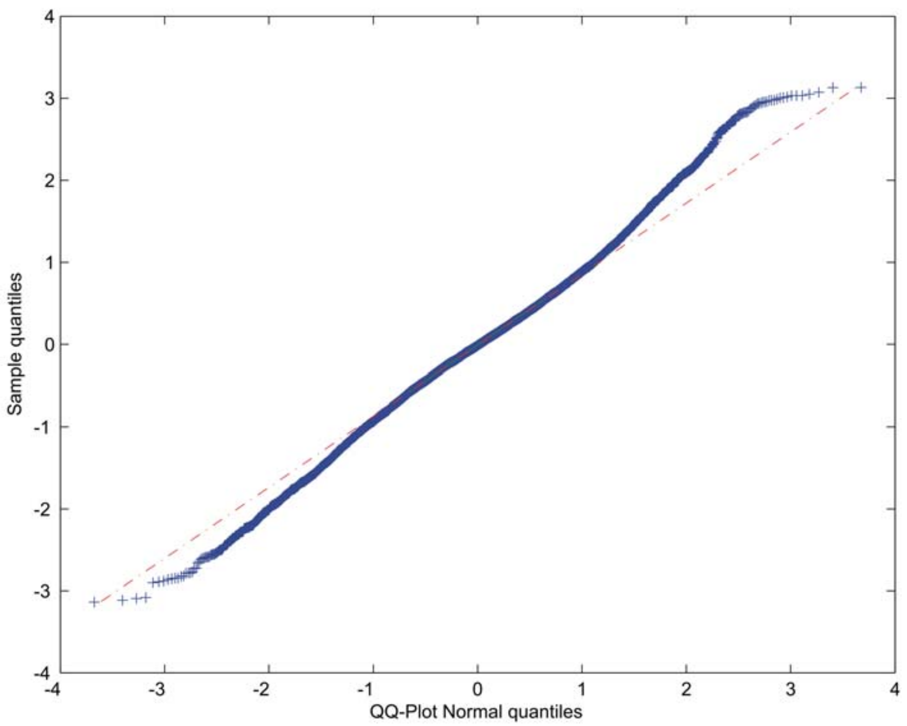
Example: Linear histogram of the L1 double difference phase observables

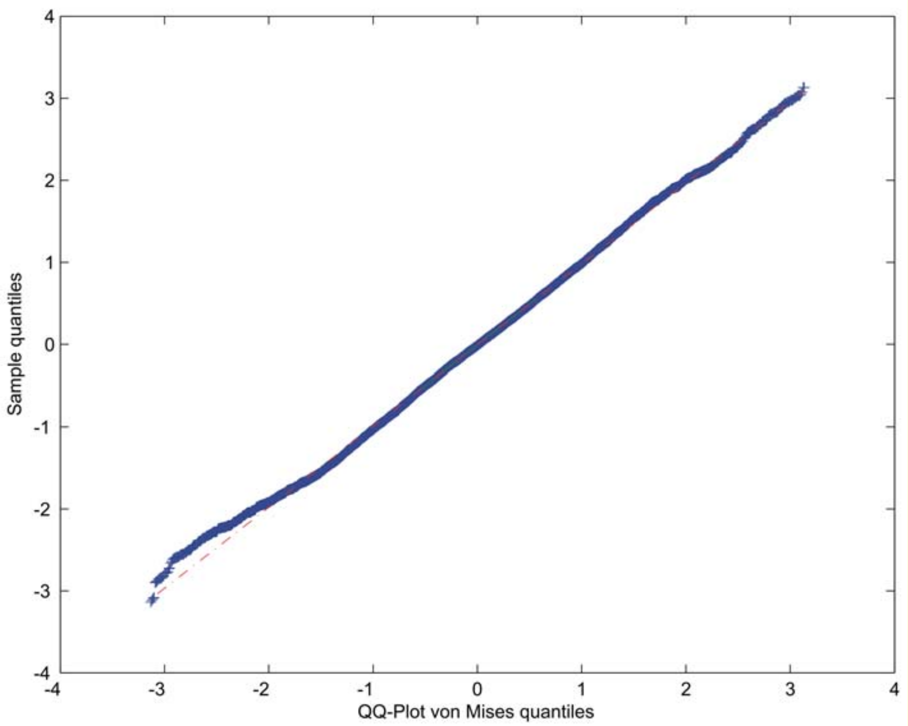


Example: Rose histogram of the L1 double difference phase observables and the mean value. (Note the arithmetic mean is +359°.34)



Example: Linear histogram of the L1 double difference phase observables and the von Mises distribution and Gauss-Laplace fits





Example: Gauss-Normal and von Mises Q-Q plots for the L1 double difference phase observables

The purpose of the quantile-quantile plot is to determine whether the sample in X is drawn from a specific (i.e., Gaussian or von Mises) distribution, or whether the samples in X and Y come from the same distribution type.

Test for goodness-of-fit:

$$H_0: \quad F = F_0, \text{ against } F \neq F_0$$

With calculation of the statistic

$$\chi^2 = \sum_{i=1}^{m} \frac{(f_i - np_i)^2}{np_i},$$

where f_i is the frequencies in interval i and p_i is the probability related certain distribution and n is the total sample number.

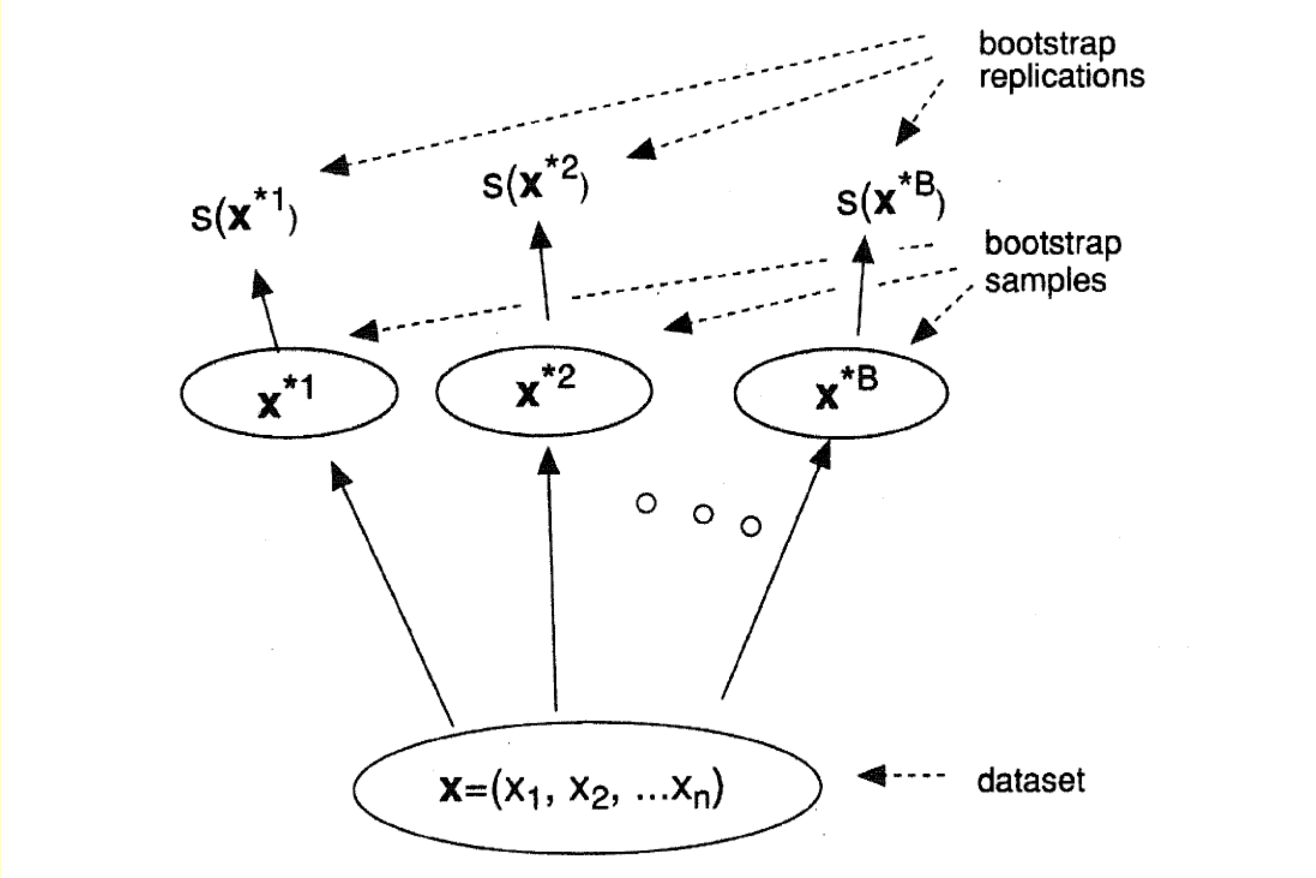
Since χ^2(VM)=59.5 is less than $\chi^2_{0.0001}(27) = 63.16$ the null hypothesis that the sample is von Mises distributed cannot be rejected.

- Indeed the close agreement between the observed and expected frequencies suggests that the von Mises distribution provides a "good fit".
- But the hypothesis of Gauss-Laplace normal is rejected since the fit results χ^2(GN)=251.4 is far greater than the critical value of 63.16.

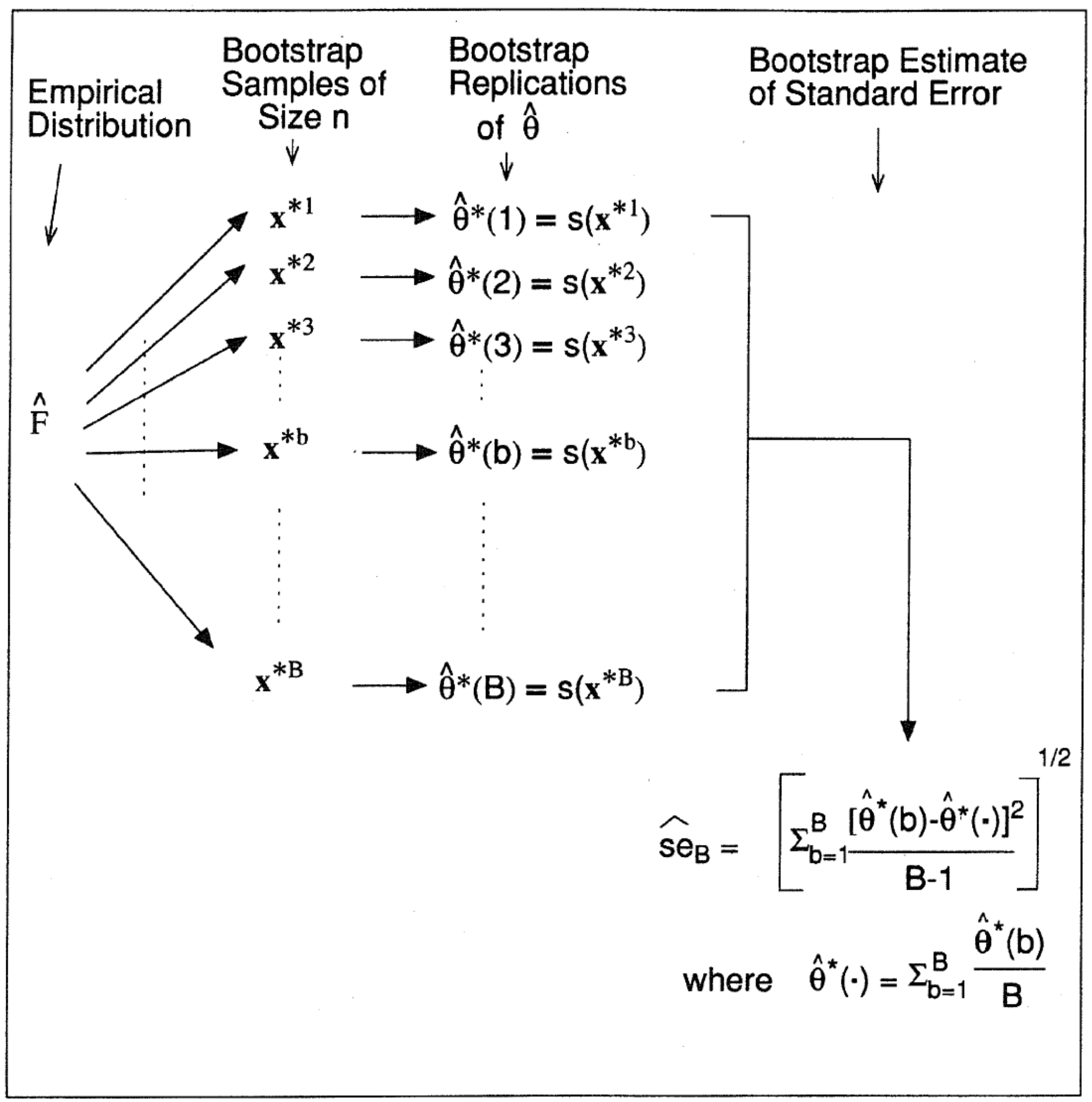
3. Bootstrap methods for the confidence domains/ hypothesis tests

■ Bootstrap methods:

- **A data-based simulation method derived from the phase *to pull oneself up by one's bootstrap*;**
- **In statistics the phase '*bootstrap method*' refers to a class of computer-intensive (resampling) statistical procedures, which is one of the modern statistical technique since 1980s;**
- **To be helpful for carrying out a statistical test or for assessing the variability of a point estimate in situations where more usual statistical procedures are not valid and /or not available (e.g. the sampling distribution of a statistic is not known);**
- **Yielding more accurate results than Gaussian approximation;**
- **One of the principal goal – to produce good confidence intervals automatically;**
- **Since the distributions of the statistics commonly used for inference on directional distributions are more complex than those arising in standard Gauss normal theory, *bootstrap methods* are particularly useful in the directional context.**



Schematic of the bootstrap process for estimating the standard error of a statistic s(x). *B* bootstrap samples are generated from the original data set. (after Efron and Tibshirani, 1993)



The bootstrap algorithm for estimating the standard error of a statistic $\hat{\theta}=s(\mathbf{x})$; each bootstrap samples is an independent random sample of size *n* from $\hat{F}$. (after Efron and Tibshirani, 1993)

■ Two distinguished Bootstrap methods:

- **Parametric *bootstrap* – a particular mathematical model is available;**
- **Nonparametric *bootstrap* – without such mathematical model.**

■ Two Bootstrap analysis methods for linear model:

- **Bootstrapping Residuals - Fit the linear model and obtain the *n* residuals:** $\mathbf{y}^* = \mathbf{G}\boldsymbol{\gamma} + \mathbf{e}$
- **Bootstrapping Pairs - Resampling on the pairs of one observable and cooresponding row of design matrix:** $\mathbf{y}^{**} = \mathbf{G}^{**}\boldsymbol{\gamma} + \mathbf{e}$

- **In the linear model context, these bootstrap methods provide inference procedures (e.g. confidence sets) that are more accurate than those produced by the other methods.**
- **Just the case for the validation and hypothesis tests of the float and fixed estimates of GPS mixed models in the directional context, with the emphasis on the determination of the confidence intervals of the estimates.**

■ Bootstrap analysis method for linear model:

- **Bootstrapping Residuals** - Fit the linear model and obtain the *n* residuals
- Choose a sample of size *n* from the residuals, generated with the probability 1/*n* for each residual, and sample with replacement. Attach these sampled values to the *n* predicted $\hat{y}_i$ to give a resampled set of *y*'s.
- Thus if the model is $\mathbf{y}=\mathbf{G}\boldsymbol{\gamma}+\mathbf{e}$ and $\hat{\mathbf{y}}=\mathbf{G}\hat{\boldsymbol{\gamma}}$ ($\hat{\boldsymbol{\gamma}}$ obtained by the LS estimator), the new bootstrapped y-values are

$$\mathbf{y}^*=\mathbf{G}\hat{\boldsymbol{\gamma}}+\mathbf{e}^*$$

where $\mathbf{e}^*$ is a resampled set from the vector $\hat{\mathbf{e}}=\mathbf{y}-\hat{\mathbf{y}}$.

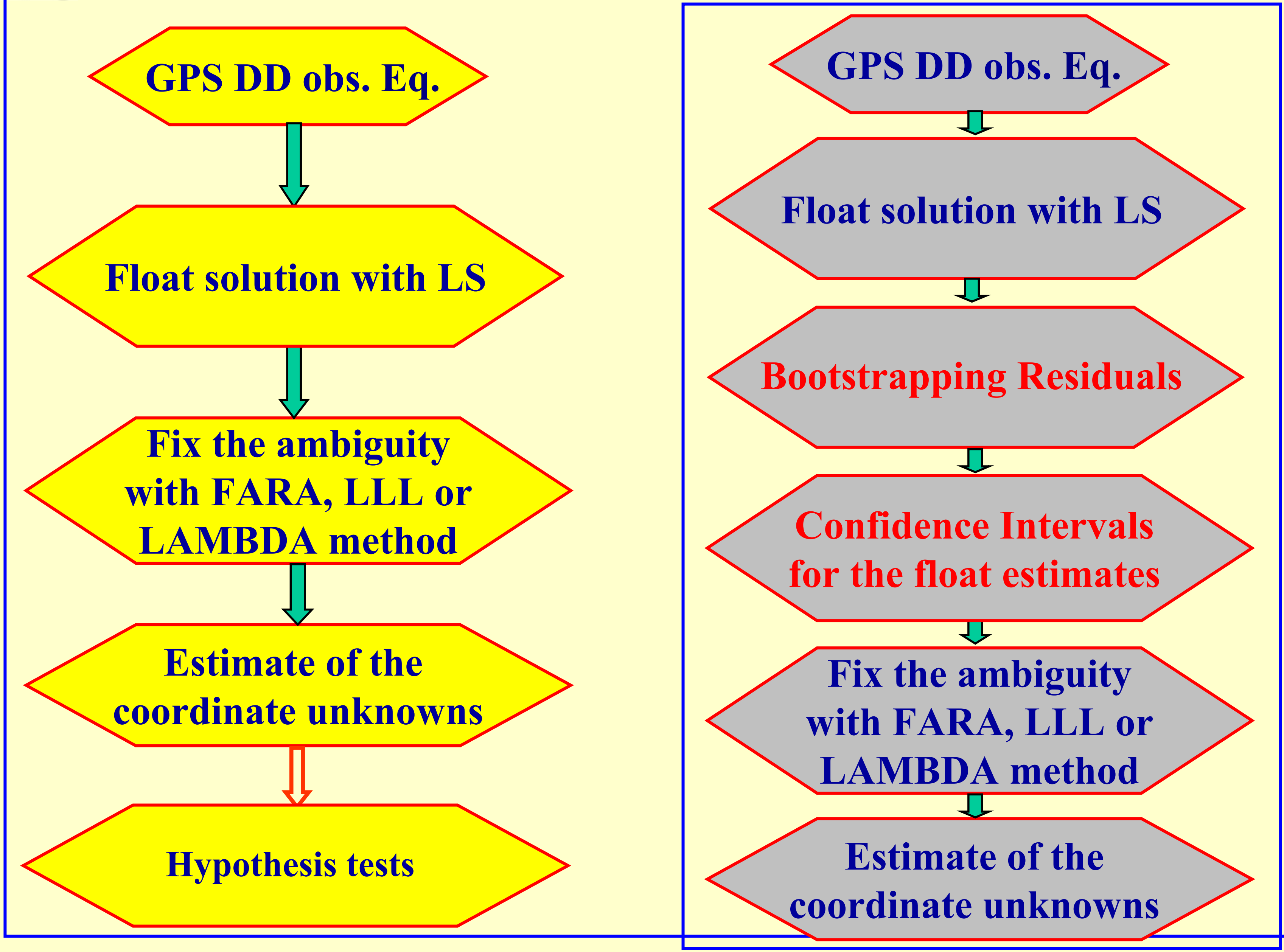
- LS estimation is now performed on the model

$$\mathbf{y}^*=\mathbf{G}\boldsymbol{\gamma}+\mathbf{e}$$

to obtain an estimate $\hat{\boldsymbol{\gamma}}^*$. As many iterations as desired can be performed, and the usual sample mean and sample standard deviation of those vector estimates can be found, which allows constructing confidence domains of the estimated parameters.

- Normally we can perform the resampling iterations with 1000 times.

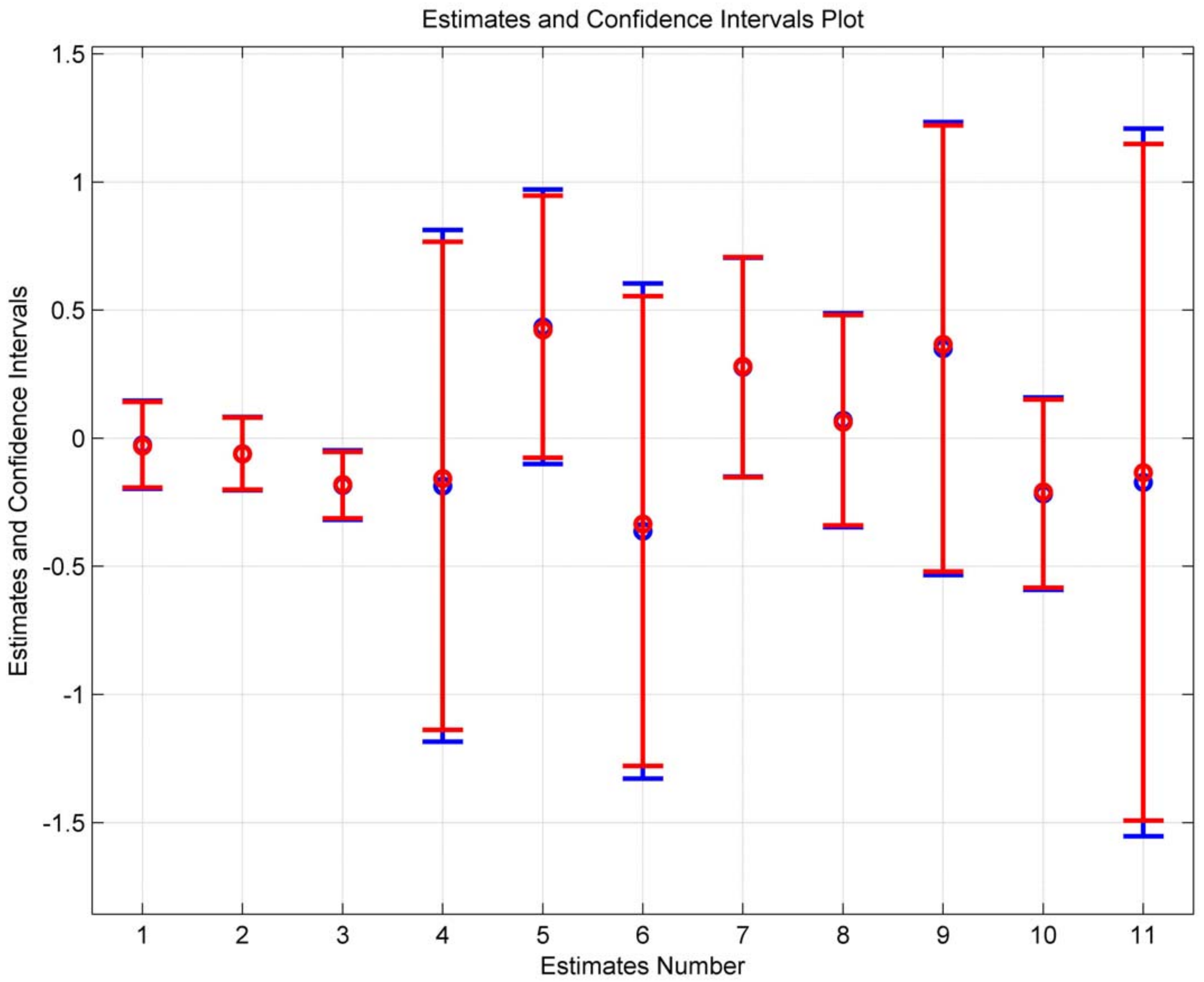
Bootstrapping confidence intervals for the float solutions



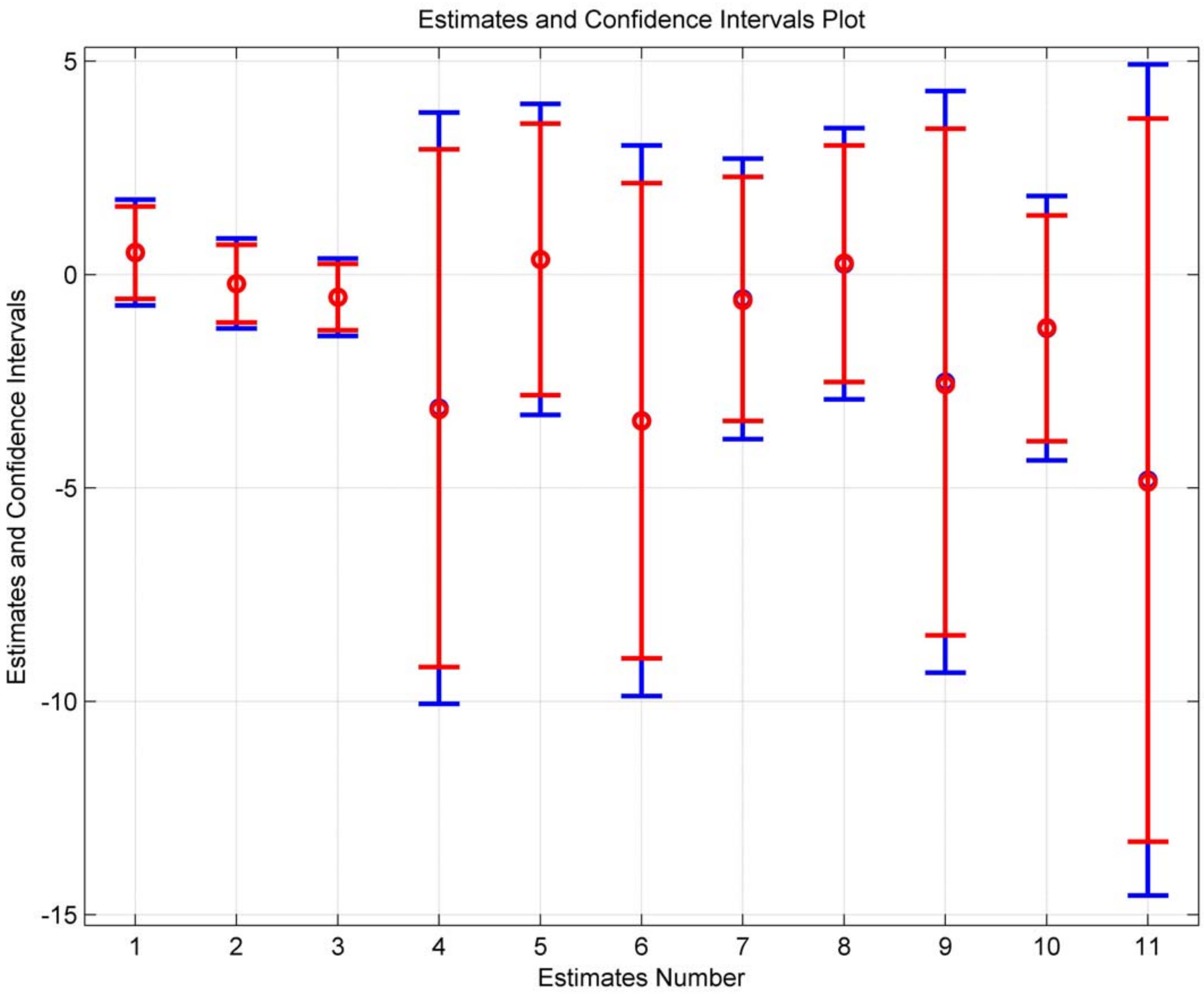
GIS

■ Testing with GPS observation set:

- **Short baselines test data: about 2 hour observations with 20 second sampling rate at one baselines (~3.6 km);**
- **Phase baseline lengths were calculated using observations above 10º;**
- **There are total 320 L1 double difference phase observables;**
- **For the testing observation period 5~20 epochs there are 11 unknown parameters, including 3 coordinate differences and 8 ambiguities.**



The comparison of the float estimates and their confidence intervals with the LS and bootstrapping residuals methods (20 epochs).



The comparison of the float estimates and their confidence intervals with the LS and bootstrapping residuals methods (5 epochs).

■ Analysis of the Bootstrapping confidence intervals for the float solutions:

- **Bootstrapping residuals for linear model provides us an efficient and accurate algorithm to construct the confidence domains of the GPS float solutions;**
- **The bootstrapping confidence intervals are consistent with the LS confidence intervals based on the t-test.**
- **Both kinds of the confidence intervals all cover the potential correct fixed ambiguity integers, which are important for searching process and fixed solution.**
- **But the bootstrapping confidence intervals are derived without any assumption about the probability distribution of the observations.**

Note: The Bootstrapped confidence sets are slightly varied among the every resampling (simulation) process.

4. Conclusion and outlook

- The statistical property of the fractional phase measurements of the GPS *double difference carrier phase* is validated as von Mises distribution;

- The classical testing theory (such as, t-test, χ^2-test, F-test and the related ratio-test) can not be simply applied to the GPS data analysis since the GPS *carrier phase observables* are not Gauss normally distributed anymore;

- We have studied the bootstrap algorithms and successfully applied the efficient bootstrapping residuals method to construct the confidence domains of the GPS float solutions;

- This answers the open question mentioned above and provides a complete solution for the estimation and hypothesis tests on the parameters of the GPS mixed integer linear models in the directional context.

Some selected References:

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- ***Thank you !***